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CRITICAL COMBINATIONS OF LONGITUDINAL AND TRANSVERSE
DIRECT STRESS FOR AN INFINITELY LONG FLAT PLATE WITH
EDGES ELASTICALLY RESTRAINED AGAINST ROTATION

By S. B. Batdorf, Manuel Stein, and Charles Libove

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

ADVANCE RESTRICTED REPORT

CRITICAL COMBINATIONS OF LONGITUDINAL AND TRANSVERSE
DIRECT STRESS FOR AN INFINITELY LONG FLAT PLATE WITH
EDGES ELASTICALLY RESTRAINED AGAINST ROTATION

By S. B. Batdorf, Manuel Stein, and Charles Libove

SUMMARY

A theoretical investigation was made of the buckling of an infinitely long flat plate with edges elastically restrained against rotation under combinations of longitudinal and transverse direct stress. Interaction curves are presented that give the critical combinations of stress for several different degrees of elastic edge restraint, including simple support and complete fixity. It was found that an appreciable fraction of the critical longitudinal stress may be applied to the plate without any reduction in the transverse compressive stress required for buckling.

INTRODUCTION

Because the skin of an airplane in flight is subjected to combinations of stress, attention has recently been given to the problem of plate buckling when more than one stress is acting. The present paper is the third of a series of papers analyzing the elastic buckling, under the action of two stresses, of an infinitely long flat plate with edges equally restrained against rotation and fully supported. The two previous papers are reference 1, which deals with the interaction of shear and longitudinal direct stress, and reference 2, which deals with the interaction of shear and transverse direct stress. The present paper describes the interaction of longitudinal and transverse direct stress. These three loading combinations are illustrated in figure 1.

Interaction curves that give the critical direct-stress combinations for several different degrees of elastic edge restraint, including simple support and complete fixity, are presented for the case in which the magnitude of the restraint is independent of the buckle wave length. These curves are based on an exact solution of the differential equation of equilibrium, the details of which are given in the appendix.

SYMBOLS

E	elastic modulus of plate material
μ	Poisson's ratio for plate material
t	thickness of plate
b	width of plate
a	length of plate ($a > b$)
D	flexural stiffness of plate per unit length
	$\left[\frac{Et^3}{12(1 - \mu^2)} \right]$
x	longitudinal coordinate
y	transverse coordinate
w	normal displacement of a point on buckled plate from its undeflected position
λ	half wave length of buckle
S_0	rotational stiffness of restraining medium along edges of plate, moment per quarter radian per unit length
c	dimensionless elastic edge-restraint constant
	$\left(\frac{4S_0 b}{D} \right)$

σ_x applied uniform longitudinal compressive stress

σ_y applied uniform transverse compressive stress

$$N_x = \sigma_x t$$

$$N_y = \sigma_y t$$

k_x, k_y dimensionless stress coefficients $\left(k_x = \frac{\sigma_x b^2 t}{\pi^2 D}; \right.$
 $\left. k_y = \frac{\sigma_y b^2 t}{\pi^2 D} \right)$

R_x longitudinal direct-stress ratio; ratio of longitudinal direct stress present to critical stress in pure longitudinal compression

R_y transverse direct-stress ratio; ratio of transverse direct stress present to critical stress in pure transverse compression

RESULTS AND DISCUSSION

The results of this investigation are given in the form of nondimensional interaction curves in figure 2. Each point on these curves represents a critical combination of the stress coefficients k_x and k_y for a given elastic edge-restraint constant ϵ at which an infinitely long flat plate will buckle. The interaction curves for plates with simply supported and clamped edges are given in figure 3 in terms of stress ratios rather than stress coefficients. The calculated data used to plot the interaction curves are given in table 1.

Applicability of the interaction curves.- Critical combinations of longitudinal and transverse direct stress for an infinitely long flat plate with edges either simply supported or clamped can be obtained from the interaction curves of figure 2. Critical combinations of direct stress for a plate with intermediate elastic restraint against edge rotation can also be obtained from figure 2 for those cases in which the stiffness of the restraining medium is independent of buckle wave length ($\epsilon = \text{a constant}$). Such edge restraint is provided

only by a medium in which rotation at one point does not influence rotation at another point. Edge conditions of this type are not ordinarily encountered but might occur when the restraint is furnished by a row of discrete elements, such as coil springs or flexible clamps. Because of the great variety of possible relationships between edge restraint and wave length, only the curves for edge restraint independent of wave length are shown. If critical stress combinations for a plate with continuous edge restraint (ϵ dependent on wave length) are desired, they can be computed - though somewhat laboriously - by the method outlined in the appendix, provided the relationship between edge restraint and wave length is known. This relationship is derived in reference 3 for the special case of a sturdy stiffener, that is, a stiffener which twists without cross-sectional distortion.

The buckling stress for a finite plate can never be lower than that for an infinite plate having the same width and thickness because the finite plate is strengthened by support along two additional edges. The use of figure 2 to estimate the critical direct stresses for a finite plate with edge restraint independent of wave length, therefore, is in all cases conservative.

Vertical portions of interaction curves.- The vertical portions of the interaction curves (fig. 2) indicate that a considerable amount of longitudinal compression may be applied to the plate without any reduction in the transverse compression required for buckling. This result parallels the result of reference 2, in which it was found that a considerable amount of shear stress could be applied to an infinitely long plate without any reduction in the transverse compression necessary to cause buckling. On the other hand, in reference 1 it was shown that the presence of shear always reduces the longitudinal compressive stress required to produce buckles. This disparity in behavior is probably attributable to the character of the buckle forms for the three types of stress. (See fig. 4.) The buckle form for shear alone (fig. 4(a)) can be transformed continuously into that for longitudinal compression alone (fig. 4(b)) by a gradual addition of compression and subtraction of shear. Neither of these buckle forms, however, can be continuously transformed into the buckle form for transverse compression alone (fig. 4(c)).

The vertical portions of the interaction curves extend indefinitely into the tension region of k_x . (For convenience, in fig. 2 the curves are stopped at a small negative value of k_x .) This property of the curves indicates that the presence of longitudinal tension has no effect upon the transverse stress necessary to produce buckling.

SUMMARY OF RESULTS

Interaction curves are presented from which critical combinations of longitudinal and transverse direct stress for an infinitely long flat plate with edges either simply supported or clamped can be obtained. Critical combinations of direct stress for intermediate elastic restraint against edge rotation can also be obtained from the interaction curves for those cases in which the stiffness of the restraining medium is independent of buckle wave length. For cases in which the stiffness of the restraining medium depends upon the buckle wave length and the relationship between the two is known, the critical combinations of direct stress can be determined - though somewhat laboriously - by a method similar to that used in obtaining the interaction curves.

A considerable amount of longitudinal compression may be applied to an infinitely long flat plate before there is any reduction in the transverse compression necessary to produce buckling. The presence of longitudinal tension has no effect upon the transverse stress necessary to produce buckling.

The use of the interaction curves to determine the critical stresses for a finite plate with edge restraint independent of wave length is in all cases conservative.

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APPENDIX

BUCKLING OF INFINITELY LONG PLATES
UNDER TWO DIRECT STRESSES

Differential equation of equilibrium.- The critical combinations of longitudinal and transverse direct stress that will cause buckling in an infinitely long flat plate with edges elastically restrained against rotation can be obtained by solving the differential equation of equilibrium. This equation, adapted from page 324 of reference 4, is

$$D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \quad (A1)$$

where N_x and N_y are positive for compression. (The coordinate system used is given in fig. 5.) Equation (A1) may be rewritten and used in the following form:

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + k_x \frac{\pi^2}{b^2} \frac{\partial^2 w}{\partial x^2} + k_y \frac{\pi^2}{b^2} \frac{\partial^2 w}{\partial y^2} = 0 \quad (A2)$$

where

$$k_x = \frac{N_x b^2}{\pi^2 D}$$

and

$$k_y = \frac{N_y b^2}{\pi^2 D}$$

Solution of differential equation.- If the plate is infinitely long in the x -direction, all displacements are periodic in x and the buckled surface is assumed to have the form

$$w = Y \cos \frac{\pi x}{\lambda} \quad (A3)$$

where Y is a function of y only and λ is the half wave length of the buckle in the x -direction. Substitution

of the expression for w given in equation (A3) in the differential equation (A2) yields the following equation:

$$\frac{d^4 y}{dy^4} - \frac{2\pi^2}{\lambda^2} \frac{d^2 y}{dy^2} + \frac{\pi^4}{\lambda^4} y - \frac{\pi^4}{b^2 \lambda^2} k_x y + \frac{\pi^2}{b^2} k_y \frac{d^2 y}{dy^2} = 0 \quad (A4)$$

Equation (A4) must be satisfied by Y if the assumed deflection is to satisfy the differential equation (A2). The expression

$$Y = e^{\frac{imy}{b}} \quad (A5)$$

will be a solution of equation (A4) when m is a root of the characteristic equation

$$m^4 + \left(2 \frac{b^2}{\lambda^2} - k_y\right) \pi^2 m^2 + \pi^4 \frac{b^2}{\lambda^2} \left(\frac{b^2}{\lambda^2} - k_x\right) = 0 \quad (A6)$$

The roots of this equation are

$$\left. \begin{aligned} m_1 &= \pi \sqrt{\frac{k_y}{2} - \frac{b^2}{\lambda^2} + \frac{1}{2} \sqrt{k_y^2 + 4 \frac{b^2}{\lambda^2} (k_x - k_y)}} \\ m_2 &= -\pi \sqrt{\frac{k_y}{2} - \frac{b^2}{\lambda^2} + \frac{1}{2} \sqrt{k_y^2 + 4 \frac{b^2}{\lambda^2} (k_x - k_y)}} \\ m_3 &= \pi \sqrt{\frac{k_y}{2} - \frac{b^2}{\lambda^2} - \frac{1}{2} \sqrt{k_y^2 + 4 \frac{b^2}{\lambda^2} (k_x - k_y)}} \\ m_4 &= -\pi \sqrt{\frac{k_y}{2} - \frac{b^2}{\lambda^2} - \frac{1}{2} \sqrt{k_y^2 + 4 \frac{b^2}{\lambda^2} (k_x - k_y)}} \end{aligned} \right\} \quad (A7)$$

The complete solution of equation (A4) is therefore

$$Y = P e^{\frac{im_1 y}{b}} + Q e^{\frac{im_2 y}{b}} + R e^{\frac{im_3 y}{b}} + S e^{\frac{im_4 y}{b}} \quad (A8)$$

where P , Q , R , and S are constants to be determined from the boundary conditions.

The solution of the differential equation (A2) can now be written

$$w = \left(P e^{\frac{i m_1 y}{b}} + Q e^{\frac{i m_2 y}{b}} + R e^{\frac{i m_3 y}{b}} + S e^{\frac{i m_4 y}{b}} \right) \cos \frac{n x}{\lambda} \quad (A9)$$

Stability criterion.— The boundary conditions that must be satisfied by the solution of the differential equation of equilibrium are

$$\left. \begin{aligned} (Y)_{y=\frac{b}{2}} &= 0 \\ (Y)_{y=-\frac{b}{2}} &= 0 \\ D\left(\frac{d^2 Y}{dy^2}\right)_{y=\frac{b}{2}} &= -4S_0\left(\frac{dY}{dy}\right)_{y=\frac{b}{2}} \\ D\left(\frac{d^2 Y}{dy^2}\right)_{y=-\frac{b}{2}} &= 4S_0\left(\frac{dY}{dy}\right)_{y=-\frac{b}{2}} \end{aligned} \right\} \quad (A10)$$

The first two conditions result from the requirement of zero deflection along the edges. The last two conditions express the requirement that the curvature at any point along the edge of the plate be consistent with the transverse bending moment at the point.

If the conditions given in equations (A10) are imposed upon equation (A8), four linear homogeneous equations in P , Q , R , and S result. These equations are

$$\left. \begin{aligned}
 &Pe \frac{im_1}{2} + Qe \frac{im_2}{2} + Re \frac{im_3}{2} + Se \frac{im_4}{2} = 0 \\
 &Pe \frac{-im_1}{2} + Qe \frac{-im_2}{2} + Re \frac{-im_3}{2} + Se \frac{-im_4}{2} = 0 \\
 &m_1^2 Pe \frac{im_1}{2} + m_2^2 Qe \frac{im_2}{2} + m_3^2 Re \frac{im_3}{2} + m_4^2 Se \frac{im_4}{2} \\
 &\quad - i\epsilon \left(m_1 Pe \frac{im_1}{2} + m_2 Qe \frac{im_2}{2} + m_3 Re \frac{im_3}{2} + m_4 Se \frac{im_4}{2} \right) = 0 \\
 &m_1^2 Pe \frac{-im_1}{2} + m_2^2 Qe \frac{-im_2}{2} + m_3^2 Re \frac{-im_3}{2} + m_4^2 Se \frac{-im_4}{2} \\
 &\quad + i\epsilon \left(m_1 Pe \frac{-im_1}{2} + m_2 Qe \frac{-im_2}{2} + m_3 Re \frac{-im_3}{2} + m_4 Se \frac{-im_4}{2} \right) = 0
 \end{aligned} \right\} \text{(All)}$$

where

$$\epsilon = \frac{4S_0 b}{D}$$

In order for P, Q, R, and S to have values other than zero, that is, in order for the plate to buckle, the determinant formed by the coefficients of P, Q, R, and S in equations (All) must equal zero. The expansion of this determinant is given on page 13 of reference 5 for the case in which the roots of the characteristic equation are of the form

$$\left. \begin{aligned}
 \frac{m_1}{2} &= \gamma + \beta \\
 \frac{m_2}{2} &= \gamma - \beta \\
 \frac{m_3}{2} &= -\gamma + i\alpha \\
 \frac{m_4}{2} &= -\gamma - i\alpha
 \end{aligned} \right\} \text{(A12)}$$

In the present problem, the roots (equations (A7)) of the characteristic equation have the form of equations (A12), where

$$\left. \begin{aligned} \gamma &= 0 \\ \beta &= \frac{\pi}{2} \sqrt{\left(\frac{k_y}{2} - \frac{b^2}{\lambda^2}\right) + \frac{1}{2} \sqrt{k_y^2 + 4 \frac{b^2}{\lambda^2} (k_x - k_y)}} \\ \alpha &= \frac{\pi}{2} \sqrt{-\left(\frac{k_y}{2} - \frac{b^2}{\lambda^2}\right) + \frac{1}{2} \sqrt{k_y^2 + 4 \frac{b^2}{\lambda^2} (k_x - k_y)}} \end{aligned} \right\} \quad (A13)$$

Substitution of $\gamma = 0$ in the stability criterion given as equation (A19) of reference 5 yields a stability criterion that is applicable to the present problem. This stability criterion is

$$\begin{aligned} & \left[(\alpha^2 + \beta^2)^2 + (\alpha^2 - \beta^2) \frac{\epsilon^2}{4} \right] \sinh 2\alpha \sin 2\beta \\ & - 2\alpha\beta \frac{\epsilon^2}{4} (\cosh 2\alpha \cos 2\beta - 1) \\ & + \epsilon \left[\alpha (\alpha^2 + \beta^2) \cosh 2\alpha \sin 2\beta - \beta (\alpha^2 + \beta^2) \sinh 2\alpha \cos 2\beta \right] = 0 \end{aligned} \quad (A14)$$

where α and β are defined in equations (A13). Any combination of values k_x , k_y , b/λ , and ϵ that satisfies equation (A14) will cause the plate to be on the point of buckling.

Interaction curves for restraint independent of wave length.— The procedure for plotting interaction curves is as follows: For a given value of ϵ , a value of k_y is chosen. Substitution of these values of k_y and ϵ in equation (A14) yields an equation in terms of k_x and b/λ . A plot of k_x against b/λ is then made. Every point on this curve represents a combination of k_x and b/λ that will maintain neutral equilibrium for the given value of ϵ and the chosen value of k_y .

Since the plate will buckle at the lowest value of k_x that will maintain neutral equilibrium, only the minimum value of k_x is taken from the plot of k_x against b/λ . This process is repeated for other assumed values of k_y , and each time a minimum value of k_x is determined. Finally, the interaction curve of k_y against the minimum value of k_x can be plotted for the given value of ϵ .

For the special case of a plate with simply supported edges ($\epsilon = 0$), equation (A14) is simplified to such an extent that the minimization of k_x with respect to b/λ can readily be done analytically. The equation of the interaction curve for $\epsilon = 0$ can then be given explicitly as

$$k_x = 2 \left(1 + \sqrt{1 - k_y} \right) \quad (A15)$$

The plotting procedure just discussed and the analytical solution for the case of simply supported edges (equation (A15)) give only the curved portions of the interaction curves. The conclusion that the vertical portions also represent critical stress combinations and are therefore properly a part of the interaction curves depends upon an argument analogous to that at the end of appendix B of reference 2. This argument is based on the fact that the end point of the curved portion of each curve can be shown to represent a combination of stresses for which the buckle wave length is infinite. When the wave length is infinite, the longitudinal stress can do no work during buckling. Accordingly, the transverse stress required to produce buckling is the same as it would be in the absence of longitudinal compression. Inasmuch as a reduction in longitudinal compression tends to increase rather than to diminish the wave length, the same argument applies to all points on a vertical line below the end point of the curved portion of each curve. For a given value of ϵ , consequently, those critical combinations of stress for which the buckle wave length is infinite are defined by a straight line of constant k_y that starts at the end point of the curved portion and extends indefinitely into the tension region of k_x . This value of k_y is the value corresponding to Euler strip buckling and is related to ϵ by the equation (adapted from equation (A21) of reference 6)

$$\epsilon = \frac{-\pi \sqrt{k_y}}{\tan \frac{\pi \sqrt{k_y}}{2}}$$

In reference 7, the problem of buckling of finite plates under combined longitudinal and transverse direct stresses is investigated. The results given in reference 7 further substantiate the existence of the vertical portions of the interaction curves, inasmuch as the finite-plate interaction curves are seen to have portions that approach vertical lines as the length-width ratio of the plate increases. In figure 6 the interaction curves for infinitely long plates with simply supported and clamped edges are compared with the curves, based on the results of reference 7, for similarly supported plates with a length-width ratio of 4.

Interaction curves for restraint dependent on wave length.- Interaction curves for a plate with edge restraint dependent on the wave length of the buckles can be obtained by a slight modification of the method outlined in the preceding section. This modification consists in computing a new value of ϵ to be used with each new assumed value of b/λ ; no other change is required. This method can be applied only when the relationship between ϵ and b/λ is known. For the special case of a sturdy stiffener, the relationship of ϵ and b/λ is derived in reference 3.

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2. Batdorf, S. B., and Houbolt, John C.: Critical Combinations of Shear and Transverse Direct Stress for an Infinitely Long Flat Plate with Edges Elastically Restrained against Rotation. NACA ARR No. L4L14, 1945.
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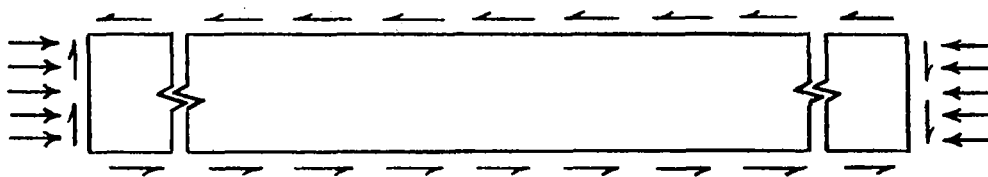
TABLE 1.- VALUES OF k_y WITH CORRESPONDING CRITICAL VALUES OF k_x

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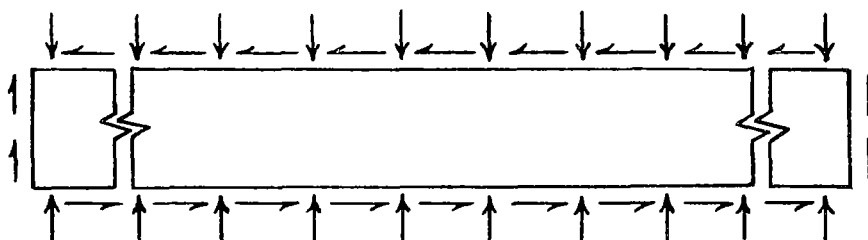
$\epsilon = 0$		$\epsilon = 2$		$\epsilon = 5$		$\epsilon = 10$		$\epsilon = \infty$	
k_y	k_x	k_y	k_x	k_y	k_x	k_y	k_x	k_y	k_x
1.00	$\begin{cases} 0 \\ 2.00 \end{cases}$	1.67	$\begin{cases} 0 \\ 2.02 \end{cases}$	2.30	$\begin{cases} 0 \\ 2.07 \end{cases}$	2.85	$\begin{cases} 0 \\ 2.17 \end{cases}$	4.00	$\begin{cases} 0 \\ 2.67 \end{cases}$
.96	2.40	1.58	2.62	2.20	2.71	2.73	2.90	3.85	3.52
.83	2.67	1.24	3.31	2.00	3.16	2.48	3.42	3.50	4.23
.75	3.00	1.03	3.62	1.71	3.62	2.13	3.92	3.00	4.85
.50	3.41	.82	3.86	1.14	4.24	1.42	4.60	2.00	5.73
.25	3.73	.41	4.26	.57	4.72	.70	5.15	1.00	6.41
0	4.00	-1.64	5.66	-2.28	6.36	0	5.60	0	6.98
-1.00	4.83	-3.28	6.47	-4.56	7.33	-2.84	6.99	-4.00	8.69
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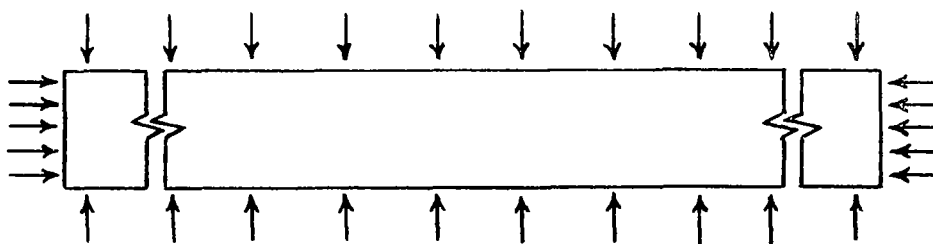
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(a) Loading combination treated in reference 1.



(b) Loading combination treated in reference 2.



(c) Loading combination treated in present paper.

Figure 1.- Buckling of an infinitely long flat plate under combined loads.

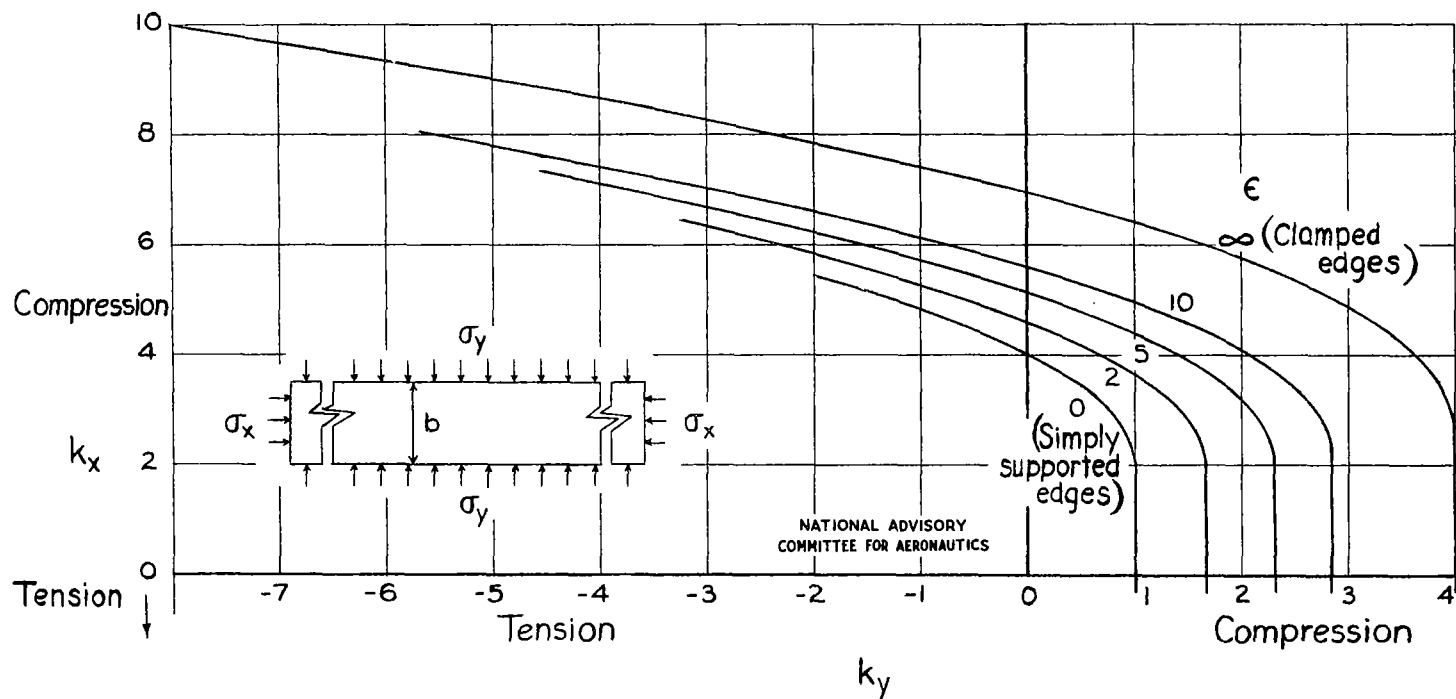


Figure 2.- Interaction curves showing critical combinations of longitudinal and transverse direct-stress coefficients for an infinitely long flat plate with edges elastically restrained against rotation.

$$\sigma_x = k_x \frac{\pi^2 E}{12(1-\mu^2)} \left(\frac{t}{b}\right)^2$$

$$\sigma_y = k_y \frac{\pi^2 E}{12(1-\mu^2)} \left(\frac{t}{b}\right)^2$$

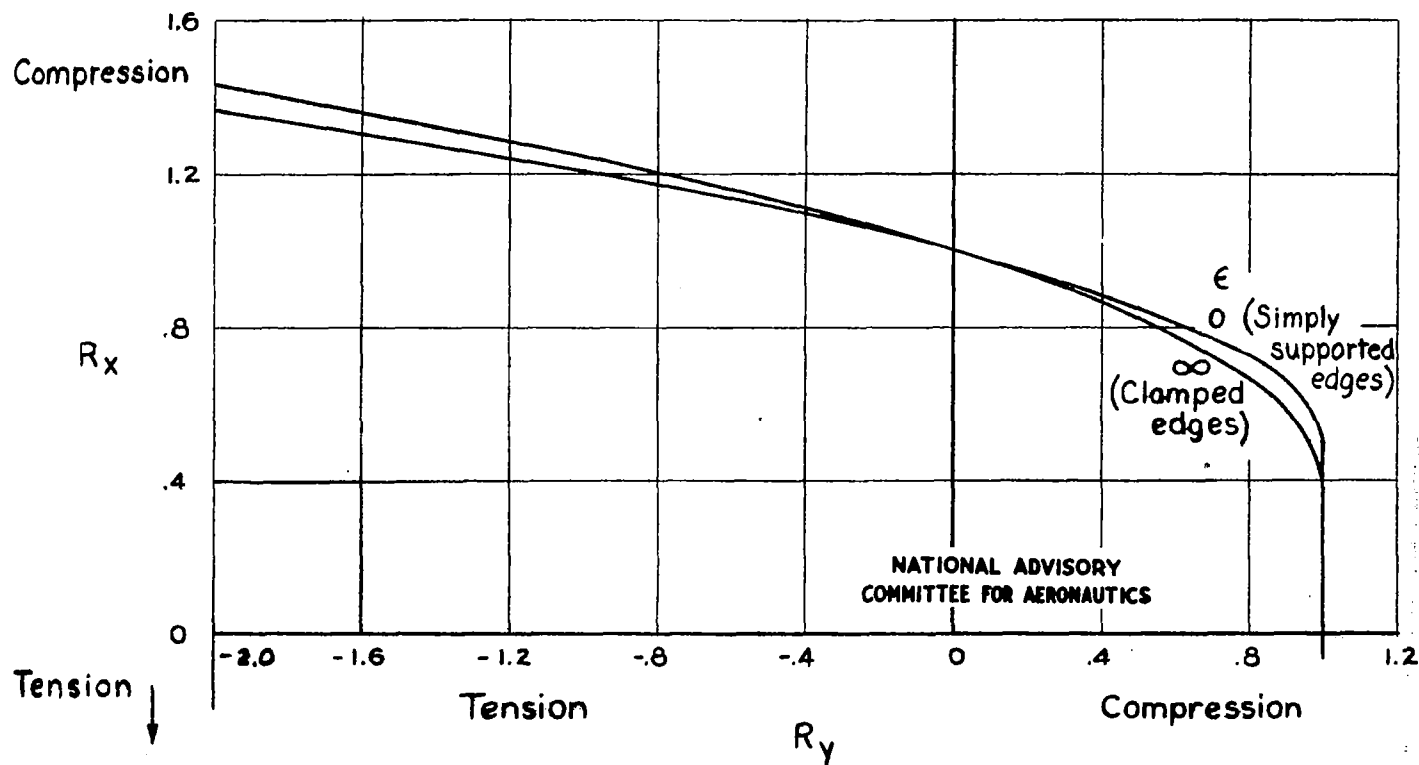
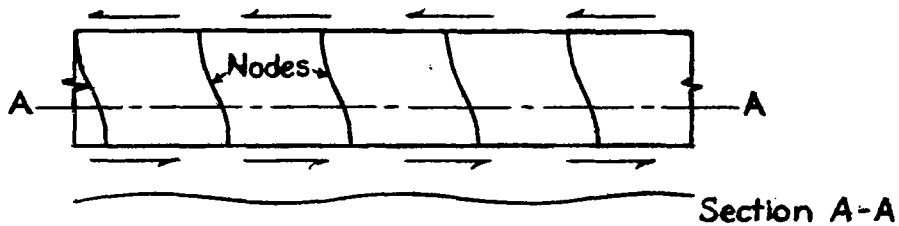
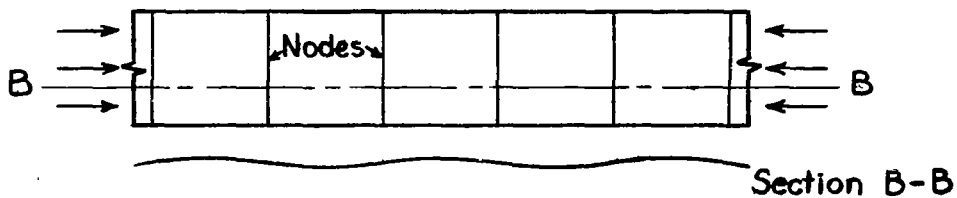


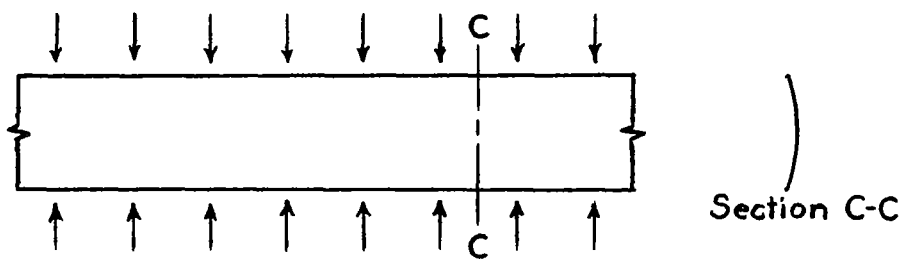
Figure 3.- Critical combinations of longitudinal direct-stress ratio R_x and transverse direct-stress ratio R_y for an infinitely long flat plate.



(a) Buckle form for shear alone.



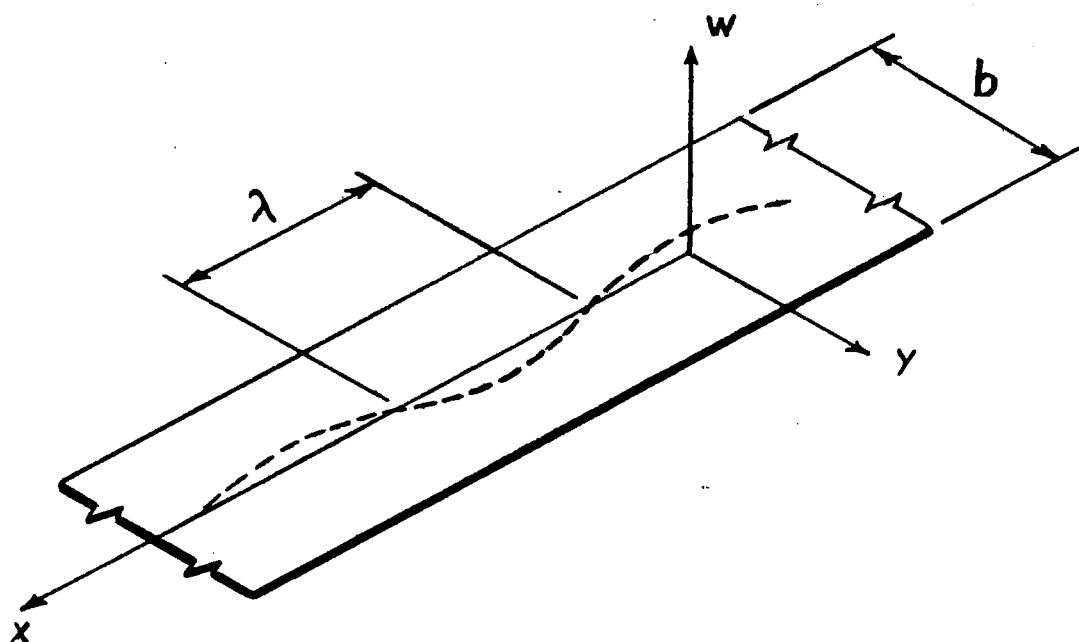
(b) Buckle form for longitudinal compression alone.



(c) Buckle form for transverse compression alone.

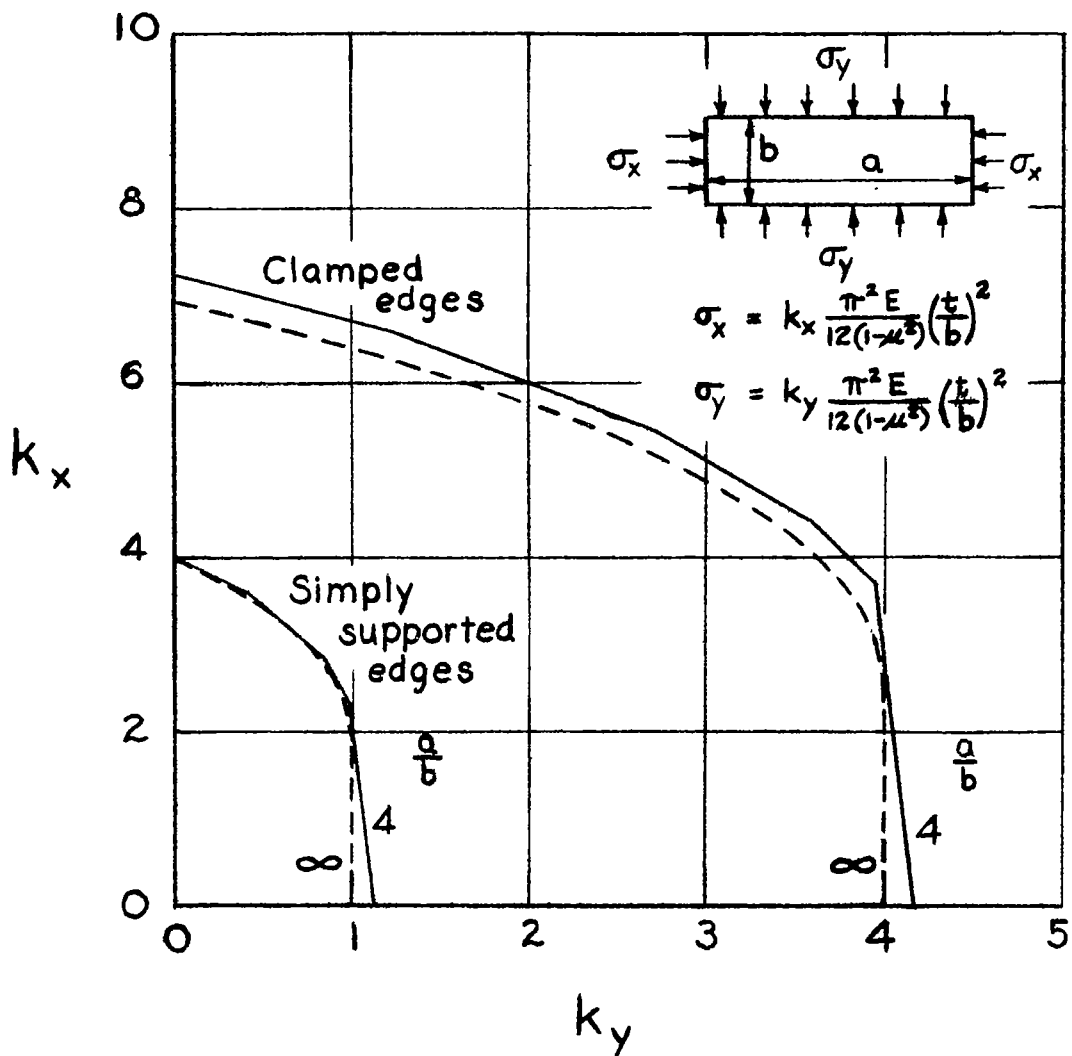
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Figure 4.- Buckle forms for a simply supported, infinitely long flat plate.



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Figure 5.- Coordinate system used in the present paper for an infinitely long flat plate.



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Figure 6.- Comparison of interaction curves for infinitely long plates and interaction curves for plates having a length-width ratio of 4.

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